Professor Dr. Jens Vygen
Matthias Kaul

Exercise Set 7

Exercise 7.1. Let G be a simple undirected graph with at least two vertices. Suppose that every vertex of G has degree at least k. Prove that there are two vertices $s, t \in V(G)$ such that G contains at least k edge-disjoint s-t-paths. Show that the statement still holds if at most one vertex has degree strictly less than k.

(4 points)

Exercise 7.2. Let G be an undirected graph. Given a partition $(X_1, \ldots, X_k)$ of $V(G)$ we define $\delta(X_1, \ldots, X_k) := \delta(X_1) \cup \cdots \cup \delta(X_k)$ (so, in particular, if $\emptyset \neq X \subsetneq V(G)$ we have $\delta(X) = \delta(X, V(G) \setminus X)$). Consider the polytope

$$R_G := \left\{ x : E(G) \to [0,1] \; : \; \sum_{e \in E(G)} x(e) = |V(G)| - 1 \text{ and } \sum_{e \in \delta(X_1, \ldots, X_k)} x(e) \geq k - 1 \text{ for every partition } (X_1, \ldots, X_k) \text{ of } V(G) \right\}$$

Show that R_G is the spanning-tree polytope of G.

(4 points)

Exercise 7.3. Let G be a graph and $T \subseteq V(G)$ with $|T|$ even. Prove:

(i) A set $F \subseteq E(G)$ intersects every T-join if and only if it contains a T-cut.

(ii) A set $F \subseteq E(G)$ intersects every T-cut if and only if it contains a T-join.

(4 points)

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Exercise 7.4. The Undirected Minimum Mean-Weight Cycle Problem is the following: Given an undirected graph G with edge-weights $c : E(G) \to \mathbb{R}$, find a cycle C whose mean-weight $c(E(C))/|E(C)|$ is minimum, or determine that G is acyclic. Consider the following algorithm for the Undirected Minimum Mean-Weight Cycle Problem: First determine with a linear search whether G has cycles or not, and if not return with this information.
Let $\gamma := \max\{c(e) : e \in E(G)\}$ and define a new edge-weight function via $c'(e) := c(e) - \gamma$. Let $T := \emptyset$. Now iterate the following: Find a minimum c'-weight T-join J with a polynomial (black-box) algorithm. If $c'(J) = 0$, return any zero-c'-weight cycle. Otherwise, let $\gamma' := c'(J)/|J|$, reset c' via $c'(e) \leftarrow c'(e) - \gamma'$, and continue.

Show that this algorithm works correctly and runs in polynomial time. Also, explain how to get the cycle to be returned in the case $c'(J) = 0$.

(6 points)

Deadline: November 26th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws24/co_exercises_ws.html`

In case of any questions feel free to contact me at mkaul@uni-bonn.de.