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Exercise Set 2

Exercise 2.1. Let G be a graph and M a matching in G that is not maximum. In this exercise we use the terminology *disjoint subgraphs/paths/circuits* and mean it quite literally: Two subgraphs are *disjoint* if they have no edges and no vertices in common. (Note that the term *vertex-disjoint paths* is often used to mean that two paths have no *inner*-vertices in common, but possibly endpoints.)

(i) Show that there are $\nu(G) - |M|$ disjoint M-augmenting paths in G.

(ii) Show the existence of an M-augmenting path of length at most $\frac{\nu(G)+|M|}{\nu(G)-|M|}$.

(iii) Let P be a shortest M-augmenting path in G and P' an $(M \Delta E(P))$-augmenting path. Prove $|E(P')| \geq |E(P)| + 2 \cdot |E(P) \cap E(P')|$.

Consider the following algorithm: We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen.

(iv) Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$, then P_i and P_j are disjoint.

(v) Show that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains less than $2\sqrt{\nu(G)} + 1$ different numbers.

From now on, let G be bipartite and set $n := |V(G)|$ and $m := |E(G)|$.

(vi) Given a non-maximum matching M in G show that we can find in $O(n+m)$-time a family $\mathcal{P}$ of disjoint shortest M-augmenting paths such that if M' is the matching obtained by augmenting M over every path in $\mathcal{P}$, then

$$\begin{aligned}\texttt{min}\{|E(P)| \,:\, P \text{ is an } M'\text{-augmenting path}\}\\ > \texttt{min}\{|E(P)| \,:\, P \text{ is an } M\text{-augmenting path}\}\end{aligned}$$

(vii) Describe an algorithm with runtime $O(\sqrt{n}(m+n))$ that solves the Cardinality Matching Problem in bipartite graphs.

(1+1+2+2+2+3+1=12 points)

Exercise 2.2. Let G be a simple, 3-regular graph.

(a) Prove that if G is bridgeless it has a perfect matching.

(b) Is there a simple, 3-regular graph without a perfect matching?

(3+1 points)

Exercise 2.3. Let G be a 3-regular undirected graph.

(a) Assume G is simple. Show that there is a matching in G covering at least $(7/8) \cdot |V(G)|$ vertices.

(b) Give an example to prove that the bound of item (a) is tight.

(c) Show that the assumption that G is simple in item (a) is necessary.

(2+1+1 points)

Deadline: October 22$^{\text{th}}$, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws24/co_exercises_ws.html`

In case of any questions feel free to contact me at mkaul@uni-bonn.de.