Exercise Set 1

Exercise 1.1. Given a directed graph G, edge capacities $u\colon E(G) \to \mathbb{R}_{\geq 0}$ and $s, t \in V(G)$, consider the linear programming formulation of the MAXIMUM FLOW PROBLEM:

$$
\begin{array}{lll}
\max & \displaystyle\sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e & \\
\text{s.t.} & \displaystyle\sum_{e \in \delta^+(v)} x_e = \sum_{e \in \delta^-(v)} x_e & \text{for all } v \in V(G) \setminus \{s, t\} \\
& x_e \leq u(e) & \text{for all } e \in E(G) \\
& x_e \geq 0 & \text{for all } e \in E(G)
\end{array}
$$

Show that the dual LP always has an integral optimum solution, and deduce the Max-Flow Min-Cut Theorem from this.

Hint: Use the complementary slackness conditions.

(4 points)

Exercise 1.2. Let G be a bipartite graph.

(a) Let $V(G) = A \,\dot\cup\, B$ be a bipartition of G.
If $A' \subseteq A$ and $B' \subseteq B$, and there are a matching $M_{A'}$ covering A' and a matching $M_{B'}$ covering B', show that there must be a matching that covers $A' \cup B'$.

(b) Suppose that for every non-empty $E' \subseteq E(G)$ we have $\tau(G - E') < \tau(G)$. Show that $E(G)$ is a matching in G.

(3+1 points)

Exercise 1.3. Let G be a k-regular bipartite graph.

(a) Prove that G contains k disjoint perfect matchings.
Hint: Use König's Theorem.

(b) Deduce from (a) that the edge set of any bipartite graph of maximum degree k can be partitioned into k matchings.

(2 + 2 points)

Exercise 1.4. Let $\alpha(G)$ denote the cardinality of a maximum stable set in G, and $\zeta(G)$ the cardinality of a minimum edge cover. Prove:

(a) $\alpha(G) + \tau(G) = |V(G)|$ for any graph G,

(b) $\nu(G) + \zeta(G) = |V(G)|$ for any graph G with no isolated vertices,

(c) $\zeta(G) = \alpha(G)$ for any bipartite graph G with no isolated vertices.

(4 points)

Submission: You may submit solutions to these exercises in groups of at most two people. Submissions should be written on physical paper, and be submitted before the start of the Tuesday lecture. To be admitted to the exam you need to collect at least half of the points across all exercise sheets, and present your solutions at least two times in class.

Deadline: October 15^{th}, before the lecture. The websites for lecture and exercises can be found at:

`https://www.or.uni-bonn.de/~vygen/lectures/cows24.html`

In case of any questions feel free to contact me at mkaul@uni-bonn.de.

Code of conduct and ombudspersons

Everyone attending this class will contribute to an inclusive and welcoming environment where we all treat each other professionally and with mutual respect, regardless of origin, beliefs, physical ability, gender or sexual identity. Discriminatory, racist, sexist, exclusionary, bullying, or harassing behavior will not be tolerated. In the event that you witness inappropriate behavior, please consider intervening if it is safe for you to do so and / or informing an ombudsperson. You might also consider contacting the victim and offering help.

The ombudspersons can be contacted at any time in the event of conflict between individuals, perceived misconduct, or any form of harassment. They are bound by secrecy. The ombudspersons in mathematics are currently:

Dr. Regula Krapf (krapf@math.uni-bonn.de), Mathematical Institute

Dr. Jack Davies (davies@math.uni-bonn.de), Mathematical Institute

For more information, see `https://www.mathematics.uni-bonn.de/en/department/fachgruppe-mathematik#fgombud`