Professor Dr. Stephan Held
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Exercise Set 2

Exercise 2.1. Let G be a graph and M a matching in G that is not maximum. In this exercise we use the terminology *disjoint subgraphs/paths/circuits* and mean it quite literally: Two subgraphs are *disjoint* if they have no edges and no vertices in common. (Note that the term *vertex-disjoint paths* is often used to mean that two paths have no *inner*-vertices in common, but possibly endpoints.)

(i) Show that there are $\nu(G) - |M|$ disjoint M-augmenting paths in G.

(ii) Show the existence of an M-augmenting path of length at most $\frac{\nu(G)+|M|}{\nu(G)-|M|}$.

(iii) Let P be a shortest M-augmenting path in G and P' an $(M \Delta E(P))$-augmenting path. Prove $|E(P')| \geq |E(P)| + 2 \cdot |E(P) \cap E(P')|$.

Consider the following algorithm: We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen.

(iv) Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$, then P_i and P_j are disjoint.

(v) Show that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains less than $2\sqrt{\nu(G)} + 1$ different numbers.

From now on, let G be bipartite and set $n := |V(G)|$ and $m := |E(G)|$.

(vi) Given a non-maximum matching M in G show that we can find in $O(n+m)$-time a family $\mathcal{P}$ of disjoint shortest M-augmenting paths such that if M' is the matching obtained by augmenting M over every path in $\mathcal{P}$, then

$$\begin{aligned}&\mathtt{min}\{|E(P)| : P \text{ is an } M'\text{-augmenting path}\}\\ &\qquad > \mathtt{min}\{|E(P)| : P \text{ is an } M\text{-augmenting path}\}\end{aligned}$$

(vii) Describe an algorithm with runtime $O(\sqrt{n}(m+n))$ that solves the Cardinality Matching Problem in bipartite graphs.

(1+1+2+2+2+3+1=12 points)

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Exercise 2.2. Let G be a 2-edge-connected graph, and let $\varphi(G)$ be the minimum number of even ears in any ear-decomposition of G. Show that then for every $v \in V(G)$ there is a matching in $G - v$ of cardinality $\frac{1}{2}\Big(n - 1 - \varphi(G)\Big)$.

(4 points)

Exercise 2.3. The *permanent* of a square matrix $M = (m_{ij})_{1 \le i,j \le n}$ is defined by

$$\mathrm{per}(M) = \sum_{\pi \in S_n} \prod_{i=1}^{n} m_{i,\pi(i)}$$

where S_n denotes the group of permutations of $\{1, \dots, n\}$ by S_n. In this exercise, you may use the following results about the permanent of M.

- If all entries of M are either 0 or 1 and its row sums are $r_1, \dots, r_n$, then $\mathrm{per}(M) \le (r_1!)^{\frac{1}{r_1}} \cdot \dots \cdot (r_n!)^{\frac{1}{r_n}}$. This was shown by Brègman[1973].
- If M is a non negative $n \times n$ matrix whose collum and row sums are all equal to 1, then $\mathrm{per}(M) \ge n! \left(\frac{1}{n}\right)^n$. This was conjectured by van der Waerden and later shown to be true by Falikman[1981] and Egoryčev[1980]. Such matrices are called *doubly stochastic matrices.*

Let G be a blanced bipartite graph on $2n$ vertices, i.e. there is a bipartition $V(G) = A \dot{\cup} B$ of G with $|A| = |B| = n$. Recall $M_G(x)$ was defined in Exercise 1.4. Finally let $\Phi(G)$ denote the number of perfect matchings in G.

(a) Prove $\Phi(G)$ and $\mathrm{per}(M_G(\mathbb{1}))$ to be equal.

(b) In the case of a k-regular G, prove $n! \left(\frac{k}{n}\right)^n \le \Phi(G) \le (k!)^{\frac{n}{k}}$.

(2 + 4 points)

Deadline: October 24th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws19/co_exercises/exercises.html`

In case of any questions feel free to contact me at rabenstein@or.uni-bonn.de.