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Exercise Set 9

Exercise 9.1. Let G be a simple undirected graph such that $|\delta(v)|$ is odd for every $v \in V(G)$. For an edge $e \in E(G)$, let $\mathcal{H}_e$ denote the family of all Hamiltonian cycles in G containing e. Show that $|\mathcal{H}_e|$ is even for every $e \in E(G)$.

(4 points)

Exercise 9.2. Let $n \geq 3$ and let $f : V(K_n) \to \mathbb{R}^2$ be an injective function. We define a function $g_f : E(K_n) \to \mathcal{P}(\mathbb{R}^2)$ by mapping an edge $\{u, v\}$ to the *open* line-segment between $f(u)$ and $f(v)$ (i.e. the line-segment between $f(u)$ and $f(v)$ without its two endpoints). We say that two edges $e, e' \in E(K_n)$ *cross* (with respect to f) if $g_f(e) \cap g_f(e') \neq \emptyset$.

Given a Hamiltonian tour T in K_n and two edges $e, e' \in E(T)$ which cross, consider the algorithm RemoveCrossing(T, e, e'): Let u, v be the endpoints of e and u', v' be the endpoints of e'. Delete e and e' from T. If $T + \{u, u'\} + \{v, v'\}$ is a Hamiltonian tour, add $\{u, u'\}$ and $\{v, v'\}$ to T. Otherwise add $\{u, v'\}$ and $\{v, u'\}$ to T.

(i) Show that after RemoveCrossing(T, e, e'), T is still a Hamiltonian tour.

Consider the algorithm RemoveAllCrossings(T): While there are edges in T that cross, choose one such pair of edges $e, e' \in E(T)$ and call RemoveCrossing(T, e, e').

(ii) Assuming that in the range of f no three points are colinear (i.e. lie on the same straight-line), show that for any Hamiltonian tour T the algorithm RemoveAllCrossings(T) terminates after $O(n^3)$ calls of RemoveCrossing.

(iii) Give an example (i.e. a choice of n, f and T) such that RemoveAllCrossings(T) does not terminate.

(1+4+1 points)

Exercise 9.3. Let $n \geq 3$ and consider the complete graph K_n with an edge-weight function $c : E(K_n) \to \mathbb{R}_{\geq 0}$. Given a (fixed) partition $(X_1, \ldots, X_k)$ of $V(K_n)$, we say that $e \in E(K_n)$ is an *intra-part edge* if $e \in E(K_n[X_1]) \,\dot\cup\, \ldots \,\dot\cup\, E(K_n[X_k])$. Otherwise, i.e. if $e \in \delta(X_1) \cup \cdots \cup \delta(X_k)$ we say e is an *inter-part edge*.

Let $(X_1, \ldots, X_k)$ be a partition of $V(K_n)$ such that every intra-part edge has weight 0.

(i) Show that every optimal solution of the TSP for (K_n, c) uses at most $k(k-1)$ edges of strictly positive weight.

(ii) Show that there is at least one optimal solution of the TSP for (K_n, c) which uses at most $k(k-1)$ inter-part edges.

(iii) Show that, using the partition $(X_1, \ldots, X_k)$, we can find an optimal solution of the TSP for (K_n, c) in $O(n^{2k(k-1)+1})$-time.

(1+1+4 points)

Deadline: December 14th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ws17/co_exercises/exercises.html`

In case of any questions feel free to contact me at silvanus@or.uni-bonn.de.