Professor Dr. Stefan Hougardy
Jannik Silvanus

Exercise Set 2

Exercise 2.1. Let G_0 be a (not necessarily simple) bipartite graph with bipartition $A \dot{\cup} B$. Assume that $|N_{G_0}(X)| \geq |X| + 1$ for all $\emptyset \neq X \subseteq A$, and for every $b \in B$, let G_b be a factor-critical graph. Let H be a graph that contains a perfect matching. Define a graph G on the vertex set

$$V(G) = A \dot{\cup} V(H) \dot{\cup} \dot{\bigcup}_{b \in B} V(G_b)$$

as follows. Keep all edges of the graphs H and G_b for $b \in B$. For every edge $\{a, b\} \in E(G_0)$ with $a \in A$, $b \in B$, add an edge connecting a to an arbitrary vertex of G_b. Add edges connecting vertices in A with vertices in $V(H)$ arbitrarily, and add arbitrary edges connecting elements of A. Let X, Y, W denote the Gallai-Edmonds decomposition of G. Show that $X = A$, $Y = \bigcup_{b \in B} V(G_b)$ and $W = V(H)$. Show that every graph arises by the above construction uniquely.

(4 points)

Exercise 2.2. Let G be a graph and M a matching in G that is not maximum. In this exercise we use the terminology *disjoint subgraphs/paths/circuits* and mean it quite literally: Two subgraphs are *disjoint* if they have no edges and no vertices in common. (Note that the term *vertex-disjoint paths* is often used to mean that two paths have no *inner*-vertices in common, but possibly endpoints.)

(i) Show that there are $\nu(G) - |M|$ disjoint M-augmenting paths in G.

(ii) Show the existence of an M-augmenting path of length at most $\frac{\nu(G)+|M|}{\nu(G)-|M|}$.

(iii) Let P be a shortest M-augmenting path in G and P' an $(M \Delta E(P))$-augmenting path. Prove $|E(P')| \geq |E(P)| + 2 \cdot |E(P) \cap E(P')|$.

Consider the following algorithm: We start with the empty matching and in each iteration augment the matching along a shortest augmenting path. Let $P_1, P_2, \ldots$ be the sequence of augmenting paths chosen.

(iv) Show that if $|E(P_i)| = |E(P_j)|$ for $i \neq j$, then P_i and P_j are disjoint.

(v) Show that the sequence $|E(P_1)|, |E(P_2)|, \ldots$ contains less than $2\sqrt{\nu(G)}+1$ different numbers.

From now on, let G be bipartite and set $n := |V(G)|$ and $m := |E(G)|$.

(vi) Given a non-maximum matching M in G show that we can find in $O(n+m)$-time a family $\mathcal{P}$ of disjoint shortest M-augmenting paths such that if M' is the matching obtained by augmenting M over every path in $\mathcal{P}$, then

$$\begin{aligned}&\mathtt{min}\{|E(P)| : P \text{ is an } M'\text{-augmenting path}\}\\ &\qquad\qquad > \mathtt{min}\{|E(P)| : P \text{ is an } M\text{-augmenting path}\}\end{aligned}$$

(vii) Describe an algorithm with runtime $O(\sqrt{n}(m+n))$ that solves the Cardinality Matching Problem in bipartite graphs.

(12 points)

Deadline: October 26th, before the lecture. The websites for lecture and exercises can be found at

`http://www.or.uni-bonn.de/lectures/ws17/coex.html`

In case of any questions feel free to contact me at silvanus@or.uni-bonn.de.