Winter term 2013/14
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Combinatorial Optimization

Exercise Sheet 7

Exercise 7.1:
Let G be a graph, $T \subseteq V(G)$ with $|T|$ even, and $F \subseteq E(G)$. Prove:

(i) F has nonempty intersection with every T-join if and only if F contains a T-cut.

(ii) F has nonempty intersection with every T-cut if and only if F contains a T-join.

(4 points)

Exercise 7.2:
Let G be a graph and $T \subseteq V(G)$. Denote by $\nu(G,T)$ the maximum cardinality of a family of pairwise disjoint T-cuts and by $\tau(G,T)$ the minimum cardinality of a T-join.

(i) Prove $\nu(G,T) \leq \tau(G,T)$.

(ii) Give an example where $\nu(G,T) < \tau(G,T)$.

(iii) Let J be a T-join. Prove: $|J| = \tau(G,T)$ if and only if $|C \cap J| \leq |C \setminus J|$ holds for every cycle C.

(iv) Let J be a T-join of minimum cardinality. Show that $\nu(G,T) = \tau(G,T)$ if and only if there exists a family of $|J|$ pairwise disjoint J-unique cuts in G. An edge set $E' \subseteq E(G)$ is called J-unique if $|E' \cap J| = 1$.

Consider the Edge-Disjoint Paths Problem: Given two graphs $G = (V,E)$ and $H = (V,F)$, decide if there exists a family $(P_f)_{f \in F}$ of edge disjoint paths in G, where $P_{\{s,t\}}$ is an s-t-path. This problem is NP-complete even if $(V, E \dot\cup F)$ is planar.

(v) Use this fact to show that it is NP-complete to decide if, for some planar graph G and some $X \subseteq V(G)$, $\nu(G,T) = \tau(G,T)$ holds.

(8 points)

Exercise 7.3:
Show that the following algorithm finds in a graph G with edge weights $w : E(G) \to \mathbb{Q}$ a cycle $C \subset E(G)$ that minimizes $\frac{w(C)}{|C|}$ in strongly polynomial time: First reduce all edge lengths by $\max\{w(e)|e \in E(G)\}$. Then find a minimum-weight $\emptyset$-join J. If $w(J) = 0$ output a cycle of length 0, otherwise add $\frac{-w(J)}{|J|}$ to all edge lengths and iterate (i.e. find again a minimum-weight $\emptyset$-join).

(4 points)

Deadline: Thursday, December 5, 2013, before the lecture.
Note: The exercise class will visit the Christmas market in Bonn on December 3rd after the exercise. Everyone is invited to attend.