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Exercise Set 10

Exercise 10.1. (a) Find, for every $m \in \mathbb{N}$, an instance of the Min-Max Resource Sharing Problem with m resources ($m = |\mathcal{R}|$) such that

$$\inf\left\{\max_{r\in\mathcal{R}} \sum_{C\in\mathcal{N}} (b_C)_r : b_C \in B_C\right\} \geq |\mathcal{R}|\lambda^*.$$

(b) Prove that for every instance of the Min-Max Resource Sharing Problem it holds

$$\inf\left\{\max_{r\in\mathcal{R}} \sum_{C\in\mathcal{N}} (b_C)_r : b_C \in B_C\right\} \leq |\mathcal{R}|\lambda^*.$$

(5+5* points)

Exercise 10.2. In this exercise we use the notation from Section 5.3.5 from the lecture notes. Let $y \in \mathbb{R}^{\mathcal{R}}_{\geq 0}$ be a price vector. Prove that there is always an arrival time solution $a(v) \in \{a_{min}(v), a_{max}(v)\}$ that satisfies

$$\sum_{r\in\mathcal{R}} usg_{v,r}(a(v))y_r = \min_{a\in[a_{min}(v),a_{max}(v)]} \sum_{r\in\mathcal{R}} usg_{v,r}(a)y_r.$$

(5 points)

Exercise 10.3. In this exercise we want to show that the randomized rounding for the Min-Max Resource Sharing Problem can be derandomized.

Let $\mathcal{N} = \{1, ..., |\mathcal{N}|\}$ and let $\mathcal{B}_i \subseteq \mathbb{R}^{\mathcal{R}}$ ($i \in \mathcal{N}$) be finite sets. Let $(x_{i,b})_{i\in\mathcal{N},b\in\mathcal{B}_i}$ be a fractional solution of the Min-Max Resource Sharing Problem with $\sum_{b\in\mathcal{B}_i} x_{i,b} = 1$ for all $i \in \mathcal{N}$.

Consider a randomized rounding $(\hat{z}_i)_{i\in\mathcal{N}}$ that arises from choosing $\hat{z}_i = b$ with probability $x_{i,b}$ independently for each $i \in \mathcal{N}$. We write

- $\lambda := \max_{r\in\mathcal{R}} \sum_{i=1}^{|\mathcal{N}|} \sum_{b\in B_i} x_{i,b} b_r$ and $\hat{\lambda} := \max_{r\in\mathcal{R}} \sum_{i=1}^{|\mathcal{N}|} (\hat{z}_i)_r$.
- For $\delta > 0$ and $z_1 \in \mathcal{B}_1, ..., z_l \in \mathcal{B}_l$ let $\Pr(\hat{\lambda} > (1+\delta)\lambda | z_1, ..., z_l)$ denote the probability that $\hat{\lambda} > (1+\delta)\lambda$ under the condition $\hat{z}_1 = z_1, ..., \hat{z}_l = z_l$.

- $\rho_r \coloneqq \max\{b_r | i \in \mathcal{N}, b \in \mathcal{B}_i, r \in \mathcal{R}\}$

We will use the following algorithm, known as Method of conditional probabilities, to round x.

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for i = 1, ..., |N| do
    Set ẑ_i := b where b ∈ B_i minimizes Pr(λ̂ > (1 + δ)λ | ẑ_1, ..., ẑ_{i-1}, b)
end for
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(a) If $\Pr(\hat{\lambda} > (1+\delta)\lambda) < 1$, show that the method of conditional probabilities returns a rounding $\hat{z}$ satisfying $\hat{\lambda} \leq (1+\delta)\lambda$.

(b) Let $F_\delta(z_1, ..., z_l) \coloneqq$

$$\sum_{r \in \mathcal{R}} \prod_{i=1,...,l} ((1+\delta)^{(z_i)_r/\rho_r}) \prod_{i=l+1}^{|\mathcal{N}|} (\sum_{b \in \mathcal{B}_i} x_{i,b}(1+\delta)^{b_r/\rho_r})(1+\delta)^{-(1+\delta)\lambda/\rho_r}$$

Show that $F_\delta(z_1, ..., z_l)$ is a pessimistic estimator for $\Pr(\hat{\lambda} > (1+\delta)\lambda | z_1, ..., z_l)$, i.e. show that the following two statements hold:

- $\Pr(\hat{\lambda} > (1+\delta)\lambda | z_1, ..., z_l) \leq F_\delta(z_1, ..., z_l)$
- $\min_{b \in \mathcal{B}_i} F_\delta(z_1, ..., z_{i-1}, b) \leq F_\delta(z_1, ..., z_{i-1})$

(c) Assume that

$$1 - \sum_{r \in \mathcal{R}} e^{-((1+\delta)\ln(1+\delta) - \delta)\lambda/\rho_r} > 0$$

Show that $F_\delta < 1$ and that the method of conditional probabilities returns a solution $\hat{z}$ satisfying $\hat{\lambda} \leq (1+\delta)\lambda$ when minimizing $F_\delta(\hat{z}_1, ..., \hat{z}_l, b)$ instead of $\Pr(\hat{\lambda} > (1+\delta)\lambda | \hat{z}_1, ..., \hat{z}_l, b)$.

(4+3+3 points)

Deadline: June 30, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss22/chipss22_ex.html`

In case of any questions feel free to contact me at blankenburg@or.uni-bonn.de.