Exercise Set 4

Exercise 4.1. For a finite set $\emptyset \neq T \subsetneq \mathbb{R}^2$ we define

$$\mathrm{BB}(T) := \max_{(x,y)\in T} x - \min_{(x,y)\in T} x + \max_{(x,y)\in T} y - \min_{(x,y)\in T} y.$$

A *Steiner tree* for T is a tree Y with $T \subseteq V(Y) \subsetneq \mathbb{R}^2$. We denote by $\mathrm{Steiner}(T)$ the length of a shortest rectilinear (i.e. edge lengths acc. to ℓ_1) Steiner tree for T. Moreover let $\mathrm{MST}(T)$ be the length of a minimum spanning tree in the complete graph on T with edge costs ℓ_1.

Prove that:

(a) $\mathrm{BB}(T) \leq \mathrm{Steiner}(T) \leq \mathrm{MST}(T)$;

(b) $\mathrm{Steiner}(T) \leq \frac{3}{2}\,\mathrm{BB}(T)$ for $|T| \leq 5$;

(c) There is no $\alpha \in \mathbb{R}$ s.t. $\mathrm{Steiner}(T) \leq \alpha\,\mathrm{BB}(T)$ for all finite $\emptyset \neq T \subset \mathbb{R}^2$.

(2 + 3 + 2 points)

Exercise 4.2. Let (M, d) be a metric space. For $n \in \mathbb{N}_{\geq 2}$ we define the Steiner ratios

$$\mathrm{SR}(M, n) := \sup_{P=\{p_1,\dots,p_n\}\subseteq M} \frac{\mathrm{MST}(P)}{\mathrm{STEINER}(P)},$$

where $\mathrm{STEINER}(P)$ denotes the length of a minimum Steiner tree connecting all points in P and $\mathrm{MST}(P)$ denotes the size of a minimum spanning tree connecting all points in P.

(a) Let $(M, d) = (\mathbb{R}^2, d)$ with $d(x, y) := ||x - y||_2$. Show that there is a $n \in \mathbb{N}_{\geq 2}$, such that $\mathrm{SR}(M, n) \geq 2/\sqrt{3}$.

(b) Let (M, d) be an arbitrary metric space and let $n \in \mathbb{N}_{\geq 2}$. Show that $\mathrm{SR}(M, n) \leq 2(1 - 1/n)$.

(2 + 3 points)

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Exercise 4.3. Let (G, c, T) be an instance of the STEINER TREE PROBLEM, G connected, $t \in T$ a terminal and $k \in \mathbb{N}$ with $k \geq 1$.
For each of the following functions $V(G) \times 2^T \to \mathbb{R}_{\geq 0}$ decide whether it defines a valid lower bound for instances of the RECTILINEAR STEINER TREE PROBLEM and prove your statement.

(a) For two valid lower bounds lb_a and lb_b, define $\max(\text{lb}_a, \text{lb}_b)$ by

$$\max(\text{lb}_a, \text{lb}_b)(v, I) := \max\Big(\text{lb}_a(v, I), \text{lb}_b(v, I)\Big).$$

(b) Define $\text{lb}_{\text{BB}}(v, I) := \text{BB}(\{v\} \cup I)$.

(c) Define $\text{lb}_{\text{mst}}(v, I) := \frac{\text{mst}(\{v\} \cup I)}{2}$. Here $\text{mst}(\{v\} \cup I)$ denotes the cost of a minimal spanning tree in $G'[\{v\} \cup I]$, where G' is the metric closure of G.

(d) Define $\text{lb}_k(v, I) := \max\Big\{\text{smt}(J) \;\Big|\; t \in J \subseteq I \cup \{v\}, |J| \leq k+1\Big\}$ if $t \in I$ and $\text{lb}_k(v, I) := 0$ otherwise.

(2 + 1 + 2 + 3 points)

Deadline: May 5, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss22/chipss22_ex.html`

In case of any questions feel free to contact me at blankenburg@or.uni-bonn.de.