Linear and Integer Programming

- Time: Tuesdays and Thursdays, 16:15 - 17:55 (with 10 minutes break)
- Place: Gerhard-Konow-Hörsaal, Lennéstr. 2
- Website: `www.or.uni-bonn.de/lectures/ss19/lgo_ss19.html`
- Lecture notes and all slides can be found on the website.

Final Examination

- Written examination
- Dates: Monday, July 15, 2019 and Saturday, September 21, 2019.
- There will be no third examination in this semester.

Exercise Classes

- Exercise classes are two hours per week.
- Assignments are released every Thursday.
- There will be programming exercises.
- 50 % of all points in the assignments are required to participate in the exam.
- Students can work in groups of two.
- All participants of a group have to be able to explain their solutions.
- Exercise classes begin in the second week.

Possible Time Slots for the Exercise Classes

1. Mo 10 - 12
2. Tu 10 - 12
3. Tu 14 - 16
4. We 12 - 14
5. We 14 - 16
6. We 16 - 18
7. Th 10 - 12
8. Th 12 - 14
9. Th 14 - 16
10. Fr 10 - 12
11. Fr 14 - 16

We will choose four of these time slots.

Application for the exercise classes: Use the form on the website

`www.or.uni-bonn.de/lectures/ss19/lgo_uebung_ss19.html`

Definition

- An **optimization problem** is a pair (I, f) with a set I and $f : I \to \mathbb{R}$.
- The elements of I are called **feasible solutions** of (I, f).
- If $I = \emptyset$, (I, f) is called **infeasible**, otherwise we call it **feasible**.
- The function f is called the **objective function** of (I, f).
- We ask for an $x^* \in I$ (called **optimum solution**) such that
  - for all $x \in I$ we have $f(x) \leq f(x^*)$ (then (I, f) is called a **maximization problem**)
  - for all $x \in I$ we have $f(x) \geq f(x^*)$ (then (I, f) is called a **minimization problem**).
- (I, f) is **unbounded** if for all $K \in \mathbb{R}$, there is an $x \in I$ with $f(x) > K$ (for the maximization problem) or an $x \in I$ with $f(x) < K$ (for the minimization problem).
- An optimization problem is called **bounded** if it is not unbounded.

LINEAR PROGRAMMING

Instance: A matrix $A \in \mathbb{R}^{m \times n}$, vectors $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

Task: Find a vector $x \in \mathbb{R}^n$ with $Ax \leq b$ maximizing $c^t x$.

Example:

$$\max \quad (3, -2, 5) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\text{s.t.} \quad \begin{pmatrix} -2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Standard Forms

Standard inequality form:

$$\begin{aligned} &\max c^t x \\ \text{s.t.} \quad & Ax \leq b \end{aligned} \tag{1}$$

Standard equational form:

$$\begin{aligned} &\max c^t x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{2}$$

Both forms can be transformed into each other.

Integer Linear Programming

Integer LINEAR PROGRAMMING

Instance: A matrix $A \in \mathbb{R}^{m \times n}$, vectors $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

Task: Find a vector $x \in \mathbb{Z}^n$ with $Ax \leq b$ maximizing $c^t x$.

If only some variables have to be integral $\Rightarrow$ MIXED INTEGER LINEAR PROGRAMMING (MILP)

Modelling Optimization Problems as LPs

Defintion

Let G be a directed graph with capacities $u : E(G) \to \mathbb{R}_{>0}$ and let s and t be two vertices of G. A feasible **s-t-flow** in (G, u) is a mapping $f : E(G) \to \mathbb{R}_{\geq 0}$ with

- $f(e) \leq u(e)$ for all $e \in E(G)$ and
- $\Delta_f(v) := \sum_{e \in \delta^+_G(v)} f(e) - \sum_{e \in \delta^-_G(v)} f(e) = 0$ for all $v \in V(G) \setminus \{s, t\}$.

The **value** of an s-t-flow f is $\text{val}(f) = \Delta_f(s)$.

Modelling Optimization Problems as LPs

> **MAXIMUM-FLOW PROBLEM**
>
> *Instance:* A directed Graph G, capacities $u : E(G) \to \mathbb{R}_{>0}$, vertices $s, t \in V(G)$ with $s \neq t$.
>
> *Task:* Find an s-t-flow $f : E(G) \to \mathbb{R}_{\geq 0}$ of maximum value.

LP-formulation:

$$
\begin{array}{lrcll}
\max & \displaystyle\sum_{e \in \delta^+_G(s)} x_e - \sum_{e \in \delta^-_G(s)} x_e & & & \\
\text{s.t.} & x_e & \geq & 0 & \text{for } e \in E(G) \\
 & x_e & \leq & u(e) & \text{for } e \in E(G) \\
 & \displaystyle\sum_{e \in \delta^+_G(v)} x_e - \sum_{e \in \delta^-_G(v)} x_e & = & 0 & \text{for } v \in V(G) \setminus \{s, t\}
\end{array}
$$

Modelling Optimization Problems as LPs

BOTTLENECK MAXIMUM-FLOW PROBLEM WITH 2 SOURCES

Instance: A directed Graph G, capacities $u : E(G) \to \mathbb{R}_{>0}$, three vertices $s_1, s_2, t \in V(G)$.

Task: Find a mapping $f : E(G) \to \mathbb{R}_{\geq 0}$ with

- $f(e) \leq u(e)$ for all $e \in E(G)$ and
- $\Delta_f(e) = 0$ for all $v \in V(G) \setminus \{s_1, s_2, t\}$

such that $\min\{\Delta_f(s_1), \Delta_f(s_2)\}$ is maximized

How can this problem be modelled by an LP?

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
& \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
& & 8x_1 & + & 12x_2 & \leq & 7 \\
& & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Goal: Find an upper bound on the optimum solution value.

Combine constraint 1 and 2:

$$12x_1 + 10x_2 = 2 \cdot (4x_1 + 2x_2) + \frac{1}{2}(8x_1 + 12x_2) \leq 2 \cdot 5 + \frac{1}{2} \cdot 7 = 13.5.$$

Combine constraint 2 and 3:

$$12x_1 + 10x_2 = \frac{7}{6} \cdot (8x_1 + 12x_2) + \frac{4}{3} \cdot (2x_1 - 3x_2) \leq \frac{7}{6} \cdot 7 + \frac{4}{3} \cdot 1 = 9.5.$$

Duality: Example

$$\begin{array}{llrcrcl} \text{(P)} & \max & 12x_1 & + & 10x_2 & & \\ & \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\ & & 8x_1 & + & 12x_2 & \leq & 7 \\ & & 2x_1 & - & 3x_2 & \leq & 1 \end{array}$$

General approach: Find numbers $u_1, u_2, u_3 \in \mathbb{R}_{\geq 0}$ such that

$$12x_1 + 10x_2 = u_1 \cdot (4x_1 + 2x_2) + u_2 \cdot (8x_1 + 12x_2) + u_3 \cdot (2x_1 - 3x_2).$$

$\Rightarrow$ $5u_1 + 7u_2 + u_3$ is an upper bound on the value of any solution of (P).

$\Rightarrow$ Chose u_1, u_2, u_3 such that $5u_1 + 7u_2 + u_3$ is minimized.

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
& \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
& & 8x_1 & + & 12x_2 & \leq & 7 \\
& & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Determine u_1, u_2, and u_3 by the following linear program:

$$
\begin{array}{llrcrcrcl}
\text{(D)} & \min & 5u_1 & + & 7u_2 & + & u_3 & & \\
& \text{s.t.} & 4u_1 & + & 8u_2 & + & 2u_3 & = & 12 \\
& & 2u_1 & + & 12u_2 & - & 3u_3 & = & 10 \\
& & u_1 & & & & & \geq & 0 \\
& & & & u_2 & & & \geq & 0 \\
& & & & & & u_3 & \geq & 0
\end{array}
$$

$\Rightarrow$ Any solution of (D) gives an upper bound for (P).

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
 & \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
 & & 8x_1 & + & 12x_2 & \leq & 7 \\
 & & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Determine u_1, u_2, and u_3 by the following linear program:

$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 5u_1 & + & 7u_2 & + & u_3 & & \\
 & \text{s.t.} & 4u_1 & + & 8u_2 & + & 2u_3 & = & 12 \\
 & & 2u_1 & + & 12u_2 & - & 3u_3 & = & 10 \\
 & & u_1 & & & & & \geq & 0 \\
 & & & & u_2 & & & \geq & 0 \\
 & & & & & & u_3 & \geq & 0
\end{array}
$$

$\Rightarrow$ Any solution of (D) gives an upper bound for (P).

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
 & \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
 & & 8x_1 & + & 12x_2 & \leq & 7 \\
 & & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Determine u_1, u_2, and u_3 by the following linear program:

$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 5u_1 & + & 7u_2 & + & 1u_3 & & \\
 & \text{s.t.} & 4u_1 & + & 8u_2 & + & 2u_3 & = & 12 \\
 & & 2u_1 & + & 12u_2 & - & 3u_3 & = & 10 \\
 & & u_1 & & & & & \geq & 0 \\
 & & & & u_2 & & & \geq & 0 \\
 & & & & & & u_3 & \geq & 0
\end{array}
$$

$\Rightarrow$ Any solution of (D) gives an upper bound for (P).

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
& \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
& & 8x_1 & + & 12x_2 & \leq & 7 \\
& & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Determine u_1, u_2, and u_3 by the following linear program:

$$
\begin{array}{llrcrcrcl}
\text{(D)} & \min & 5u_1 & + & 7u_2 & + & u_3 & & \\
& \text{s.t.} & 4u_1 & + & 8u_2 & + & 2u_3 & = & 12 \\
& & 2u_1 & + & 12u_2 & - & 3u_3 & = & 10 \\
& & u_1 & & & & & \geq & 0 \\
& & & & u_2 & & & \geq & 0 \\
& & & & & & u_3 & \geq & 0
\end{array}
$$

$\Rightarrow$ Any solution of (D) gives an upper bound for (P).

Duality: Example

$$
\begin{array}{llrcrcl}
\text{(P)} & \max & 12x_1 & + & 10x_2 & & \\
& \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 5 \\
& & 8x_1 & + & 12x_2 & \leq & 7 \\
& & 2x_1 & - & 3x_2 & \leq & 1
\end{array}
$$

Determine u_1, u_2, and u_3 by the following linear program:

$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 5u_1 & + & 7u_2 & + & u_3 & & \\
& \text{s.t.} & 4u_1 & + & 8u_2 & + & 2u_3 & = & 12 \\
& & 2u_1 & + & 12u_2 & - & 3u_3 & = & 10 \\
& & u_1 & & & & & \geq & 0 \\
& & & & u_2 & & & \geq & 0 \\
& & & & & & u_3 & \geq & 0
\end{array}
$$

$\Rightarrow$ Any solution of (D) gives an upper bound for (P).

Fourier-Motzkin Elimination I

Given a system of inequalities, check if a solution exists.

$$
\begin{array}{rcrcrcr}
3x & + & 2y & + & 4z & \leq & 10 \\
3x & & & + & 2z & \leq & 9 \\
2x & - & y & & & \leq & 5 \\
-x & + & 2y & - & z & \leq & 3 \\
-2x & & & & & \leq & 4 \\
& & 2y & + & 2z & \leq & 7
\end{array}
$$

First step: Get rid of variable x.

Fourier-Motzkin Elimination II

$$\begin{array}{rcrcrcr}
3x & + & 2y & + & 4z & \leq & 10 \\
3x & & & + & 2z & \leq & 9 \\
2x & - & y & & & \leq & 5 \\
-x & + & 2y & - & z & \leq & 3 \\
-2x & & & & & \leq & 4 \\
 & & 2y & + & 2z & \leq & 7
\end{array}$$

is equivalent to

$$\begin{array}{rcrcrcr}
x & \leq & \frac{10}{3} & - & \frac{2}{3}y & - & \frac{4}{3}z \\
x & \leq & 3 & & & - & \frac{2}{3}z \\
x & \leq & \frac{5}{2} & + & \frac{1}{2}y & & \\
x & \geq & -3 & + & 2y & - & z \\
x & \geq & -2 & & & & \\
 & & 2y & + & 2z & \leq & 7
\end{array}$$

Fourier-Motzkin Elimination III

$$
\begin{array}{ccccccccc}
x & \leq & \frac{10}{3} & - & \frac{2}{3}y & - & \frac{4}{3}z \\
x & \leq & 3 & & & - & \frac{2}{3}z \\
x & \leq & \frac{5}{2} & + & \frac{1}{2}y & & \\
x & \geq & -3 & + & 2y & - & z \\
x & \geq & -2 & & & & \\
 & & 2y & + & 2z & \leq & 7
\end{array}
$$

This system is feasible if and only if the following system has a solution:

$$
\begin{array}{rcl}
\min\left\{\frac{10}{3} - \frac{2}{3}y - \frac{4}{3}z, \quad 3 - \frac{2}{3}z, \quad \frac{5}{2} + \frac{1}{2}y\right\} & \geq & \max\{-3 + 2y - z, \quad -2\} \\
2y + 2z & \leq & 7
\end{array}
$$

Fourier-Motzkin Elimination IV

$$\min\left\{\tfrac{10}{3} - \tfrac{2}{3}y - \tfrac{4}{3}z, \quad 3 - \tfrac{2}{3}z, \quad \tfrac{5}{2} + \tfrac{1}{2}y\right\} \geq \max\{-3 + 2y - z, \quad -2\}$$
$$2y + 2z \leq 7$$

This system can be rewritten in the following way:

$$\begin{array}{rcrcrcrcrcr}
\frac{10}{3} & - & \frac{2}{3}y & - & \frac{4}{3}z & \geq & -3 & + & 2y & - & z \\
\frac{10}{3} & - & \frac{2}{3}y & - & \frac{4}{3}z & \geq & -2 & & & & \\
3 & & & - & \frac{2}{3}z & \geq & -3 & + & 2y & - & z \\
3 & & & - & \frac{2}{3}z & \geq & -2 & & & & \\
\frac{5}{2} & + & \frac{1}{2}y & & & \geq & -3 & + & 2y & - & z \\
\frac{5}{2} & + & \frac{1}{2}y & & & \geq & -2 & & & & \\
 & & 2y & + & 2z & \leq & 7 & & & &
\end{array}$$

Fourier-Motzkin Elimination V

Conversion in standard form:

$$\begin{array}{rcrcr}
\frac{8}{3}y & + & \frac{1}{3}z & \leq & \frac{19}{3} \\
\frac{2}{3}y & + & \frac{4}{3}z & \leq & \frac{16}{3} \\
\frac{8}{3}y & - & z & \leq & 6 \\
 & & \frac{2}{3}z & \leq & 5 \\
\frac{3}{2}y & - & z & \leq & \frac{11}{2} \\
-\frac{1}{2}y & & & \leq & \frac{9}{2} \\
2y & + & 2z & \leq & 7
\end{array}$$

Iterate these steps and remove *all* variables.

Farkas' Lemma

Theorem (Farkas' Lemma, most general case)

For $A \in \mathbb{R}^{m_1 \times n_1}$, $B \in \mathbb{R}^{m_1 \times n_2}$, $C \in \mathbb{R}^{m_2 \times n_1}$, $D \in \mathbb{R}^{m_2 \times n_2}$, $a \in \mathbb{R}^{m_1}$ and $b \in \mathbb{R}^{m_2}$ exactly one of the two following systems has a feasible solution:

System 1:

$$\begin{array}{rcrcl} Ax & + & By & \leq & a \\ Cx & + & Dy & = & b \\ x & & & \geq & 0 \end{array}$$

System 2:

$$\begin{array}{rcrcl} u^tA & + & v^tC & \geq & 0^t \\ u^tB & + & v^tD & = & 0^t \\ u & & & \geq & 0 \\ u^ta & + & v^tb & < & 0 \end{array}$$

Corollary

Let A,B,C,D,E,F,G,H,K be matrices and a,b,c,d,e,f be vectors of appropriate dimensions such that:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix} \text{ is an } m \times n\text{-matrix,}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is a vector of length } m \text{ and } \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ is a vector of length } n. \text{ Then}$$

$$\max \left\{ d^t x + e^t y + f^t z \quad : \quad \begin{array}{ccccccc} Ax & + & By & + & Cz & \leq & a \\ Dx & + & Ey & + & Fz & = & b \\ Gx & + & Hy & + & Kz & \geq & c \\ x & & & & & \geq & 0 \\ & & & & z & \leq & 0 \end{array} \right\}$$

$$=$$

$$\min \left\{ a^t u + b^t v + c^t w \quad : \quad \begin{array}{ccccccc} A^t u & + & D^t v & + & G^t w & \geq & d \\ B^t u & + & E^t v & + & H^t w & = & e \\ C^t u & + & F^t v & + & K^t w & \leq & f \\ u & & & & & \geq & 0 \\ & & & & w & \leq & 0 \end{array} \right\},$$

provided that both sets are non-empty.

Corollary

Let A,B,C,D,E,F,G,H,K be matrices and a,b,c,d,e,f be vectors of appropriate dimensions such that:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix} \text{ is an } m \times n\text{-matrix,}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is a vector of length } m \text{ and } \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ is a vector of length } n. \text{ Then}$$

$$\max \left\{ d^t x + e^t y + f^t z \;:\; \begin{array}{ccccccc} Ax & + & By & + & Cz & \leq & a \\ Dx & + & Ey & + & Fz & = & b \\ Gx & + & Hy & + & Kz & \geq & c \\ x & & & & & \geq & 0 \\ & & & & z & \leq & 0 \end{array} \right\}$$

$$=$$

$$\min \left\{ a^t u + b^t v + c^t w \;:\; \begin{array}{ccccccc} A^t u & + & D^t v & + & G^t w & \geq & d \\ B^t u & + & E^t v & + & H^t w & = & e \\ C^t u & + & F^t v & + & K^t w & \leq & f \\ u & & & & & \geq & 0 \\ & & & & w & \leq & 0 \end{array} \right\},$$

provided that both sets are non-empty.

Corollary

Let A,B,C,D,E,F,G,H,K be matrices and a,b,c,d,e,f be vectors of appropriate dimensions such that:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix} \text{ is an } m \times n\text{-matrix,}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is a vector of length } m \text{ and } \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ is a vector of length } n. \text{ Then}$$

$$\max \left\{ d^t x + e^t y + f^t z \quad : \quad \begin{array}{ccccccc} Ax & + & By & + & Cz & \leq & a \\ Dx & + & Ey & + & Fz & = & b \\ Gx & + & Hy & + & Kz & \geq & c \\ x & & & & & \geq & 0 \\ & & & & z & \leq & 0 \end{array} \right\}$$

$$=$$

$$\min \left\{ a^t u + b^t v + c^t w \quad : \quad \begin{array}{ccccccc} A^t u & + & D^t v & + & G^t w & \geq & d \\ B^t u & + & E^t v & + & H^t w & = & e \\ C^t u & + & F^t v & + & K^t w & \leq & f \\ u & & & & & \geq & 0 \\ & & & & w & \leq & 0 \end{array} \right\},$$

provided that both sets are non-empty.

Corollary

Let A,B,C,D,E,F,G,H,K be matrices and a,b,c,d,e,f be vectors of appropriate dimensions such that:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix} \text{ is an } m \times n\text{-matrix,}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is a vector of length } m \text{ and } \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ is a vector of length } n. \text{ Then}$$

$$\max \left\{ d^t x + e^t y + f^t z \;:\; \begin{array}{ccccccc} Ax & + & By & + & Cz & \leq & a \\ Dx & + & Ey & + & Fz & = & b \\ Gx & + & Hy & + & Kz & \geq & c \\ x & & & & & \geq & 0 \\ & & & & z & \leq & 0 \end{array} \right\}$$

$$=$$

$$\min \left\{ a^t u + b^t v + c^t w \;:\; \begin{array}{ccccccc} A^t u & + & D^t v & + & G^t w & \geq & d \\ B^t u & + & E^t v & + & H^t w & = & e \\ C^t u & + & F^t v & + & K^t w & \leq & f \\ u & & & & & \geq & 0 \\ & & & & w & \leq & 0 \end{array} \right\},$$

provided that both sets are non-empty.

Corollary

Let A,B,C,D,E,F,G,H,K be matrices and a,b,c,d,e,f be vectors of appropriate dimensions such that:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix} \text{ is an } m \times n\text{-matrix,}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is a vector of length } m \text{ and } \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ is a vector of length } n. \text{ Then}$$

$$\max \left\{ d^t x + e^t y + f^t z \;:\; \begin{array}{ccccccc} Ax & + & By & + & Cz & \leq & a \\ Dx & + & Ey & + & Fz & = & b \\ Gx & + & Hy & + & Kz & \geq & c \\ x & & & & & \geq & 0 \\ & & & & z & \leq & 0 \end{array} \right\}$$

$$=$$

$$\min \left\{ a^t u + b^t v + c^t w \;:\; \begin{array}{ccccccc} A^t u & + & D^t v & + & G^t w & \geq & d \\ B^t u & + & E^t v & + & H^t w & = & e \\ C^t u & + & F^t v & + & K^t w & \leq & f \\ u & & & & & \geq & 0 \\ & & & & w & \leq & 0 \end{array} \right\},$$

provided that both sets are non-empty.

Theorem (Strict Complementary Slackness)

Let $\max\{c^t x \mid Ax \leq b\}$ and $\min\{b^t y \mid A^t y = c, y \geq 0\}$ be a pair of a primal and a dual linear program that are both feasible. Then, for each inequality $a_i^t x \leq b_i$ in $Ax \leq b$ exactly one of the following two statements holds:

(a) The primal LP $\max\{c^t x \mid Ax \leq b\}$ has an optimum solution x^* with $a_i^t x^* < b_i$.

(b) The dual LP $\min\{b^t y \mid A^t y = c, y \geq 0\}$ has an optimum solution y^* with $y_i^* > 0$.

Max-Flow Problem

Assumption: No edges enter s or leave t.

LP-formulation:

$$
\begin{array}{llrcll}
\max & & \displaystyle\sum_{e \in \delta^+_G(s)} x_e & & & \\
\text{s.t.} & & x_e & \geq & 0 & \text{for } e \in E(G) \\
& & x_e & \leq & u(e) & \text{for } e \in E(G) \\
& \displaystyle\sum_{e \in \delta^+_G(v)} x_e - & \displaystyle\sum_{e \in \delta^-_G(v)} x_e & = & 0 & \text{for } v \in V(G) \setminus \{s, t\}
\end{array}
$$

Dual LP:

$$
\begin{array}{llcll}
\min & \displaystyle\sum_{e \in E(G)} u(e) y_e & & & \\
\text{s.t.} & y_e & \geq & 0 & \text{for } e \in E(G) \\
& y_e + z_v - z_w & \geq & 0 & \text{for } e = (v, w) \in E(G), \{s, t\} \cap \{v, w\} = \emptyset \\
& y_e + z_v & \geq & 0 & \text{for } e = (v, t) \in E(G), v \neq s \\
& y_e - z_w & \geq & 1 & \text{for } e = (s, w) \in E(G), w \neq t \\
& y_e & \geq & 1 & \text{for } e = (s, t) \in E(G)
\end{array}
$$

Max-Flow Problem

Assumption: No edges enter s or leave t.

LP-formulation:

$$
\begin{array}{llcll}
\max & \displaystyle\sum_{e\in\delta^+_G(s)} x_e & & & \\
\text{s.t.} & x_e & \geq & 0 & \text{for } e \in E(G) \\
 & x_e & \leq & u(e) & \text{for } e \in E(G) \\
 & \displaystyle\sum_{e\in\delta^+_G(v)} x_e - \sum_{e\in\delta^-_G(v)} x_e & = & 0 & \text{for } v \in V(G) \setminus \{s,t\}
\end{array}
$$

Dual LP (simplified):

$$
\begin{array}{llcll}
\min & \displaystyle\sum_{e\in E(G)} u(e)y_e & & & \\
\text{s.t.} & y_e & \geq & 0 & \text{for } e \in E(G) \\
 & y_e + z_v - z_w & \geq & 0 & \text{for } e = (v,w) \in E(G) \\
 & z_s & = & -1 & \\
 & z_t & = & 0 &
\end{array}
$$

Max-Flow Problem

Assumption: No edges enter s or leave t.

LP-formulation:

$$\begin{array}{llcll}
\max & \sum_{e \in \delta_G^+(s)} x_e & & & \\
\text{s.t.} & x_e & \geq & 0 & \text{for } e \in E(G) \\
& x_e & \leq & u(e) & \text{for } e \in E(G) \\
& \sum_{e \in \delta_G^+(v)} x_e - \sum_{e \in \delta_G^-(v)} x_e & = & 0 & \text{for } v \in V(G) \setminus \{s, t\}
\end{array}$$

Dual LP:

$$\begin{array}{llcll}
\min & \sum_{e \in E(G)} u(e) y_e & & & \\
\text{s.t.} & y_e & \geq & 0 & \text{for } e \in E(G) \\
& y_e + z_v - z_w & \geq & 0 & \text{for } e = (v, w) \in E(G) \\
& z_s & = & -1 & \\
& z_t & = & 0 &
\end{array}$$

Proposition

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polyhedron and $F \subseteq P$. Then, the following statements are equivalent:

(a) F is a face of P.

(b) There is a vector $c \in \mathbb{R}^n$ such that $\delta := \max\{c^t x \mid x \in P\} < \infty$ and $F = \{x \in P \mid c^t x = \delta\}$.

(c) There is a subsystem $A'x \leq b'$ of $Ax \leq b$ such that $F = \{x \in P \mid A'x = b'\} \neq \emptyset$.

Simplex Algorithm: Example I

$$
\begin{array}{lrcrcrcrcrcl}
\max & x_1 & + & x_2 & & & & & & & & \\
s.t. & -x_1 & + & x_2 & + & x_3 & & & & & = & 1 \\
 & x_1 & & & & & + & x_4 & & & = & 3 \\
 & & & x_2 & & & & & + & x_5 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0
\end{array}
$$

Initial basis: $\{3, 4, 5\}$. $\Rightarrow$ $A_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Simplex tableau:

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}
$$

Recent solution: $(0, 0, 1, 3, 2)$

Simplex Algorithm: Example I

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}
$$

Increase exactly one of the non-basic variables with positive coefficient in the objective function.
We choose x_2. How much can we increase it?

Constraints:
$x_3 = 1 + x_1 - x_2$: x_2 cannot get larger than 1.
$x_4 = 3 - x_1$: no constraint on x_2.
$x_5 = 2 - x_2$: x_2 cannot get larger than 2.

Strictest constraint: $x_3 = 1 + x_1 - x_2$
$\Rightarrow$ Replace 3 by 2 in B.

Simplex Algorithm: Example I

First tableau:

$$\begin{array}{rcrcrcr} x_3 & = & 1 & + & x_1 & - & x_2 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 2 & & & - & x_2 \\ \hline z & = & & & x_1 & + & x_2 \end{array}$$

Replace 3 by 2 in the basis B: $B = \{2, 4, 5\}$:

$x_2 = 1 + x_1 - x_3$.

Second tableau:

$$\begin{array}{rcrcrcr} x_2 & = & 1 & + & x_1 & - & x_3 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 1 & - & x_1 & + & x_3 \\ \hline z & = & 1 & + & 2x_1 & - & x_3 \end{array}$$

Recent solution: $(0, 1, 0, 3, 1)$

Simplex Algorithm: Example I

Second tableau:

$$
\begin{array}{rcrcrcr}
x_2 & = & 1 & + & x_1 & - & x_3 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 1 & - & x_1 & + & x_3 \\
\hline
z & = & 1 & + & 2x_1 & - & x_3
\end{array}
$$

Only one candidate: x_1

$x_5 = 1 - x_1 + x_3$ is critical. Replace 5 by 1 in B: $B = \{1, 2, 4\}$.

$x_1 = 1 + x_3 - x_5$.

Third tableau:

$$
\begin{array}{rcrcrcr}
x_1 & = & 1 & + & x_3 & - & x_5 \\
x_2 & = & 2 & & & - & x_5 \\
x_4 & = & 2 & - & x_3 & + & x_5 \\
\hline
z & = & 3 & + & x_3 & - & 2x_5
\end{array}
$$

Recent solution: $x = (1, 2, 0, 2, 0)$.

Simplex Algorithm: Example I

Third tableau:

$$\begin{array}{rclllll} x_1 & = & 1 & + & x_3 & - & x_5 \\ x_2 & = & 2 & & & - & x_5 \\ x_4 & = & 2 & - & x_3 & + & x_5 \\ \hline z & = & 3 & + & x_3 & - & 2x_5 \end{array}$$

Only one candidate: x_3

$x_4 = 2 - x_3 + x_5$ is critical. Replace 4 by 3 in B: $B = \{1, 2, 3\}$.

$x_3 = 2 - x_4 + x_5$

Fourth tableau:

$$\begin{array}{rclllll} x_1 & = & 3 & - & x_4 & & \\ x_2 & = & 2 & & & - & x_5 \\ x_3 & = & 2 & - & x_4 & + & x_5 \\ \hline z & = & 5 & - & x_4 & - & x_5 \end{array}$$

Recent solution: $x = (3, 2, 2, 0, 0)$.

Simplex Algorithm: Example I

Fourth tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 3 & - & x_4 & & \\ x_2 & = & 2 & & & - & x_5 \\ x_3 & = & 2 & - & x_4 & + & x_5 \\ \hline z & = & 5 & - & x_4 & - & x_5 \end{array}$$

Recent solution: $x = (3, 2, 2, 0, 0)$.

This is an optimum solution!

Second Example: Unboundedness

Simplex Algorithm: Example II: Unboundedness

$$\begin{array}{lrcrcrcrcl}
\max & x_1 & & & & & & & & \\
s.t. & x_1 & - & x_2 & + & x_3 & & & = & 1 \\
 & -x_1 & + & x_2 & & & + & x_4 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}$$

Initial basis: B={3,4}

Simplex Tableau:

$$\begin{array}{rcrcrcr}
x_3 & = & 1 & - & x_1 & + & x_2 \\
x_4 & = & 2 & + & x_1 & - & x_2 \\
\hline
z & = & & & x_1 & &
\end{array}$$

Recent solution: $x = (0, 0, 1, 2)$.

Simplex Algorithm: Example II: Unboundedness

First Tableau:

$$\begin{array}{rcrcrcr} x_3 & = & 1 & - & x_1 & + & x_2 \\ x_4 & = & 2 & + & x_1 & - & x_2 \\ \hline z & = & & & x_1 & & \end{array}$$

Only one candidate: x_1. $x_3 = 1 - x_1 + x_2$ is critical. Replace 3 by 1 in B: $B = \{1, 4\}$.

$x_1 = 1 + x_2 - x_3$.

Second Tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 1 & + & x_2 & - & x_3 \\ x_4 & = & 3 & & & - & x_3 \\ \hline z & = & 1 & + & x_2 & - & x_3 \end{array}$$

Recent solution:

$x = (1, 0, 0, 3)$.

Simplex Algorithm: Example II: Unboundedness

Second Tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 1 & + & x_2 & - & x_3 \\ x_4 & = & 3 & & & - & x_3 \\ \hline z & = & 1 & + & x_2 & - & x_3 \end{array}$$

Only one candidate: x_2. No constraint for it!

$\Rightarrow$ The LP is unbounded

Second Example: Degeneracy

Simplex Algorithm: Example III: Degeneracy

$$
\begin{array}{llllllll}
\max & & & x_2 & & & & \\
s.t. & -x_1 & + & x_2 & + & x_3 & & & = & 0 \\
 & x_1 & & & & & + & x_4 & = & 2 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Initial basis: $B = \{3, 4\}$

Simplex Tableau:

$$
\begin{array}{ccccccc}
x_3 & = & & & x_1 & - & x_2 \\
x_4 & = & 2 & - & x_1 & & \\
\hline
z & = & & & & & x_2
\end{array}
$$

$\Rightarrow x = (0, 0, 0, 2)$: degenerated solution.

Simplex Algorithm: Example III: Degeneracy

First Tableau:

$$
\begin{array}{rcrcrcr}
x_3 & = & & & x_1 & - & x_2 \\
x_4 & = & 2 & - & x_1 & & \\
\hline
z & = & & & & & x_2
\end{array}
$$

Want to increase x_2. $x_3 = x_1 - x_2$ is critical. Replace 3 by 2 in B:
$B = \{2, 4\}$.
$x_2 = x_1 - x_3$. We will replace 3 by 2 in the basis.
But: We cannot increase x_2.
Second Tableau:

$$
\begin{array}{rcrcrcr}
x_2 & = & & & x_1 & - & x_3 \\
x_4 & = & 2 & - & x_1 & & \\
\hline
z & = & & & x_1 & - & x_3
\end{array}
$$

Recent solution: $x = (0, 0, 0, 2)$.

Simplex Algorithm: Example III: Degeneracy

Second Tableau:

$$\begin{array}{rcrcrcr} x_2 & = & & & x_1 & - & x_3 \\ x_4 & = & 2 & - & x_1 & & \\ \hline z & = & & & x_1 & - & x_3 \end{array}$$

Increase x_1. $x_4 = 2 - x_1$ is critical. $x_1 = 2 - x_4$. New base $B = \{1, 2, 0, 0\}$.

Third Tableau:

$$\begin{array}{rcrcrcr} x_1 & = & 2 & & & - & x_4 \\ x_2 & = & 2 & - & x_3 & - & x_4 \\ \hline z & = & 2 & - & x_3 & - & x_4 \end{array}$$

Optimum solution: $x = (2, 2, 0, 0)$.

The Simplex Algorithm

Algorithm 1: Simplex Algorithm

Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$

Output: $\tilde{x} \in \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ maximizing $c^t x$ or the message that $\max\{c^t x \mid Ax = b, x \geq 0\}$ is unbounded or infeasible

1 Compute a feasible basis B;

2 If no such basis exists, stop with the message "INFEASIBLE";

3 Set $N = \{1, \ldots, n\} \setminus B$ and compute the feasible basic solution x for B;

4 Compute the simplex tableau $\begin{array}{cccc} x_B & = & p & + & Qx_N \\ \hline z & = & z_0 & + & r^t x_N \end{array}$ for B;

5 **if** $r \leq 0$ **then**
    return $\tilde{x} = x$;

6 Choose $\alpha \in N$ with $r_\alpha > 0$;

7 **if** $q_{i\alpha} \geq 0$ **for all** $i \in B$ **then**
    return "UNBOUNDED";

8 Choose $\beta \in B$ with $q_{\beta\alpha} < 0$ and $\frac{p_\beta}{q_{\beta\alpha}} = \max\{\frac{p_i}{q_{i\alpha}} \mid q_{i\alpha} < 0, i \in B\}$;

9 Set $B = (B \setminus \{\beta\}) \cup \{\alpha\}$;

10 GOTO line 3;

The Simplex Algorithm

Algorithm 2: Simplex Algorithm

Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$

Output: $\tilde{x} \in \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ maximizing $c^t x$ or the message that $\max\{c^t x \mid Ax = b, x \geq 0\}$ is unbounded or infeasible

1 Compute a feasible basis B;

2 If no such basis exists, stop with the message "INFEASIBLE";

3 Set $N = \{1, \ldots, n\} \setminus B$ and compute the feasible basic solution x for B;

4 Compute the simplex tableau $\begin{array}{cccc} x_B & = & p & + & Qx_N \\ \hline z & = & z_0 & + & r^t x_N \end{array}$ for B;

5 **if** $r \leq 0$ **then**

  return $\tilde{x} = x$;

6 Choose $\alpha \in N$ with $r_\alpha > 0$;

7 **if** $q_{i\alpha} \geq 0$ **for all** $i \in B$ **then**

  return "UNBOUNDED";

8 Choose $\beta \in B$ with $q_{\beta\alpha} < 0$ and $\frac{p_\beta}{q_{\beta\alpha}} = \max\{\frac{p_i}{q_{i\alpha}} \mid q_{i\alpha} < 0, i \in B\}$;

9 Set $B = (B \setminus \{\beta\}) \cup \{\alpha\}$;

10 GOTO line 3;

The Simplex Algorithm

Algorithm 3: Simplex Algorithm

Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$

Output: $\tilde{x} \in \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ maximizing $c^t x$ or the message that $\max\{c^t x \mid Ax = b, x \geq 0\}$ is unbounded or infeasible

1 Compute a feasible basis B;

2 If no such basis exists, stop with the message "INFEASIBLE";

3 Set $N = \{1, \ldots, n\} \setminus B$ and compute the feasible basic solution x for B;

4 Compute the simplex tableau $\begin{array}{ccccc} x_B & = & p & + & Qx_N \\ \hline z & = & z_0 & + & r^t x_N \end{array}$ for B;

5 **if** $r \leq 0$ **then**
  return $\tilde{x} = x$;

6 Choose $\alpha \in N$ with $r_\alpha > 0$;

7 **if** $q_{i\alpha} \geq 0$ **for all** $i \in B$ **then**
  return "UNBOUNDED";

8 Choose $\beta \in B$ with $q_{\beta\alpha} < 0$ and $\frac{p_\beta}{q_{\beta\alpha}} = \max\{\frac{p_i}{q_{i\alpha}} \mid q_{i\alpha} < 0, i \in B\}$;

9 Set $B = (B \setminus \{\beta\}) \cup \{\alpha\}$;

10 GOTO line 3;

Definition

Let G be a directed graph with capacities $u: E(G) \to \mathbb{R}_{>0}$ and numbers $b: V(G) \to \mathbb{R}$ with $\sum_{v \in V(G)} b(v) = 0$. A **feasible b-flow in (G, u, b)** is a mapping $f: E(G) \to \mathbb{R}_{\geq 0}$ with

- $f(e) \leq u(e)$ for all $e \in E(G)$ and
- $\sum_{e \in \delta^+_G(v)} f(e) - \sum_{e \in \delta^-_G(v)} f(e) = b(v)$ for all $v \in V(G)$.

Notation:

- $b(v)$: **balance** of v.
- If $b(v) > 0$, we call it the **supply** of v.
- If $b(v) < 0$, we call it the **demand** of v.
- Nodes v of G with $b(v) > 0$ are called **sources**.
- Nodes v with $b(v) < 0$ are called **sinks**.

Minimum-Cost Flow Problem

- **Input:** A directed graph G, capacities $u : E(G) \to \mathbb{R}_{>0}$, numbers $b : V(G) \to \mathbb{R}$ with $\sum_{v \in V(G)} b(v) = 0$, edge costs $c : E(G) \to \mathbb{R}$.
- **Task:** Find a b-flow f minimizing $\sum_{e \in E(G)} c(e) \cdot f(e)$.

Definition

Let G be a directed graph.

- For $e = (v, w)$ let $\overleftarrow{e} = (w, v)$ its **reverse edge**.
- Define $\overleftrightarrow{G}$ by $V(\overleftrightarrow{G}) = V(G)$ and $E(\overleftrightarrow{G}) = E(G) \dot\cup \{\overleftarrow{e} \mid e \in E(G)\}$.
- Edge costs $c : E(G) \to \mathbb{R}$ are extended to $E(\overleftrightarrow{G})$ by $c(\overleftarrow{e}) := -c(e)$.
- Let (G, u, b, c) be a MINIMUM-COST FLOW instance and let f be a b-flow in (G, u). The **residual graph** $G_{u,f}$ is defined by $V(G_{u,f}) := V(G)$ and
$$E(G_{u,f}) := \{e \in E(G) \mid f(e) < u(e)\} \quad \dot\cup \quad \{\overleftarrow{e} \in E(\overleftrightarrow{G}) \mid f(e) > 0\}.$$
- For $e \in E(G)$ we define the **residual capacity** by $u_f(e) = u(e) - f(e)$ and by $u_f(\overleftarrow{e}) = f(e)$.

Augmenting Flow

If P is a subgraph of the residual graph $G_{u,f}$ then **augmenting** f **along** P **by** γ (for $\gamma > 0$) means increasing P on forward edges in P (i.e. edges in $E(G) \cap E(P)$) by γ and reducing it on reverse edges in P by γ.

Definition

Let (G, u, b, c) be a MINIMUM-COST FLOW instance with G connected. A **spanning tree structure** is a quadruple (r, T, L, U) where $r \in V(G)$, $E(G) = T \dot{\cup} L \dot{\cup} U$, $|T| = |V(G)| - 1$, and $(V(G), T)$ does not contain an undirected cycle. The **b-flow f associated to** (r, T, L, U) is defined by

- $f(e) = 0$ for $e \in L$,
- $f(e) = u(e)$ for $e \in U$,
- $f(e) = \sum_{v \in C_e} b(v) + \sum_{e' \in U \cap \delta^-(C_e)} u(e') - \sum_{e' \in U \cap \delta^+(C_e)} u(e')$ for $e \in E(T)$ where C_e is vertex set of the the connected component for $(V(G), T \setminus \{e\})$ containing v (for $e = (v, w)$).

The structure (r, T, L, U) is called **feasible** if $0 \leq f(e) \leq u(e)$ for all $e \in E(T)$. An edge $(v, w) \in E(T)$ is called **downward** if v is on the undirected r-w-path in T, otherwise is is called **upward**.

Definition

A feasible spanning tree structure (r, T, L, U) is **strongly feasible** if $0 < f(e)$ for every downward edge $e \in E(T)$ and $f(e) < u(e)$ for every upward edge $e \in E(T)$.

Definition

Let (r, T, L, U) be a spanning tree structure. The unique function $\pi : V(G) \to \mathbb{R}$ with $\pi(r) = 0$ and $c_\pi(e) := c(e) + \pi(v) - \pi(w) = 0$ for all $e = (v, w) \in T$ is called the **potential associated to** (r, T, L, U).

Algorithm 4: Network Simplex Algorithm

Input: A MIN-COST-FLOW instance (G, u, b, c);
A strongly feasible spanning tree structure (r, T, L, U).

Output: A minimum-cost flow f.

1 Compute b-flow f and potential π associated to (r, T, L, U);
2 $e_0 :=$ an edge with $e_0 \in L$ and $c_\pi(e_0) < 0$ or with $e_0 \in U$ and $c_\pi(e_0) > 0$;
3 **if** (No such edge exists) **then return** f
4 $C :=$ the fund. circuit of e_0 (if $e_0 \in L$) or of $\overleftarrow{e_0}$ (if $e_0 \in U$) and let $\rho = c_\pi(e_0)$;
5 $\gamma := \min_{e' \in E(C)} u_f(e')$.
6 $e' :=$ last edge on C with $u_f(e') = \gamma$ when C is traversed starting at the peak;
7 Let e_1 be the corresponding edge in G, i.e. $e' = e_1$ or $e' = \overleftarrow{e_1}$;
8 Remove e_0 from L or U;
9 Set $T = (T \cup \{e_0\}) \setminus \{e_1\}$;
10 **if** $e' = e_1$ **then** Set $U = U \cup \{e_1\}$;
11 **else** Set $L = L \cup \{e_1\}$;
12 Augment f along γ by C;
13 Let X be the connected component of $(V(G), T \setminus \{e_0\})$ that contains r;
14 **if** $e_0 \in \delta^+(X)$ **then** Set $\pi(v) = \pi(v) + \rho$ for $v \in V(G) \setminus X$;
15 **if** $e_0 \in \delta^-(X)$ **then** Set $\pi(v) = \pi(v) - \rho$ for $v \in V(G) \setminus X$;
16 **go to** line 2;

Illustration:

T :

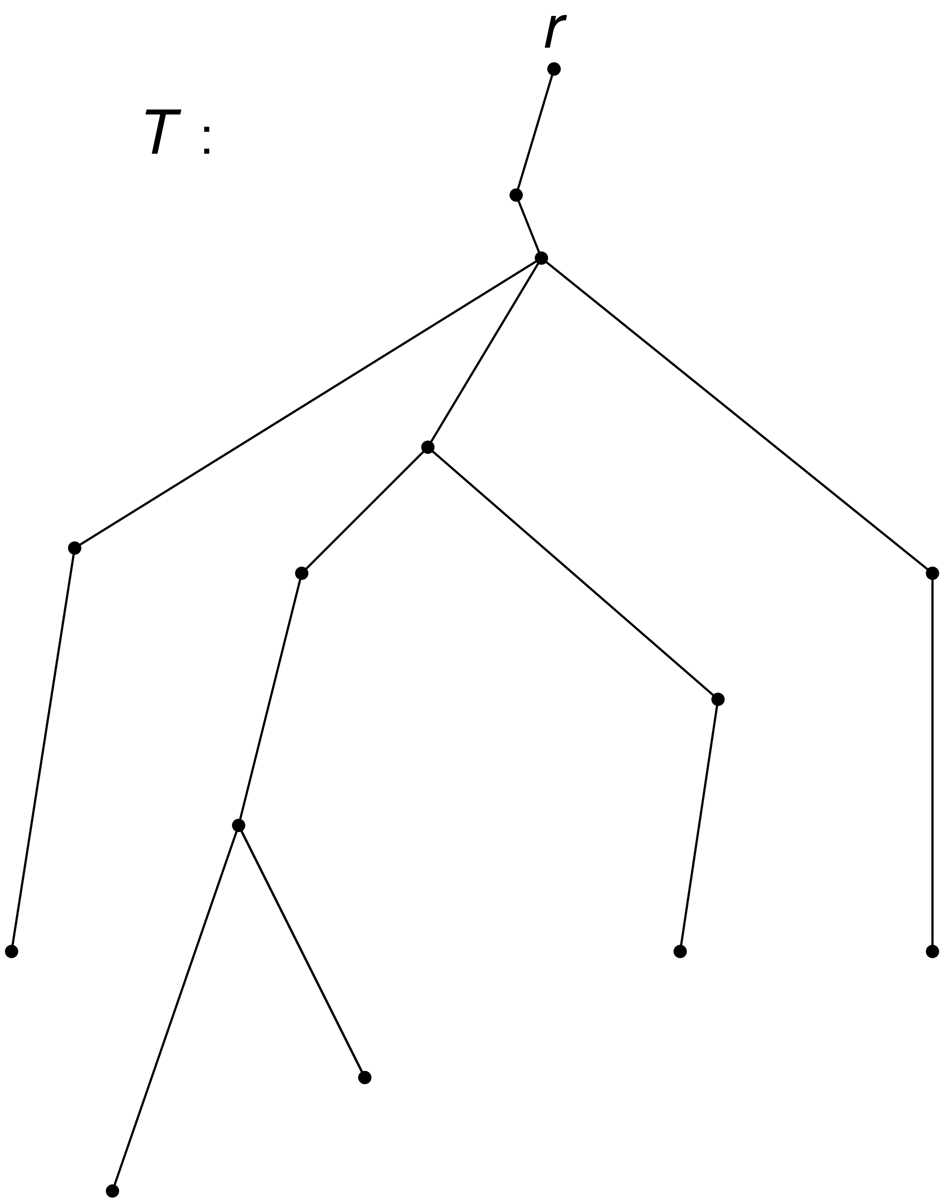
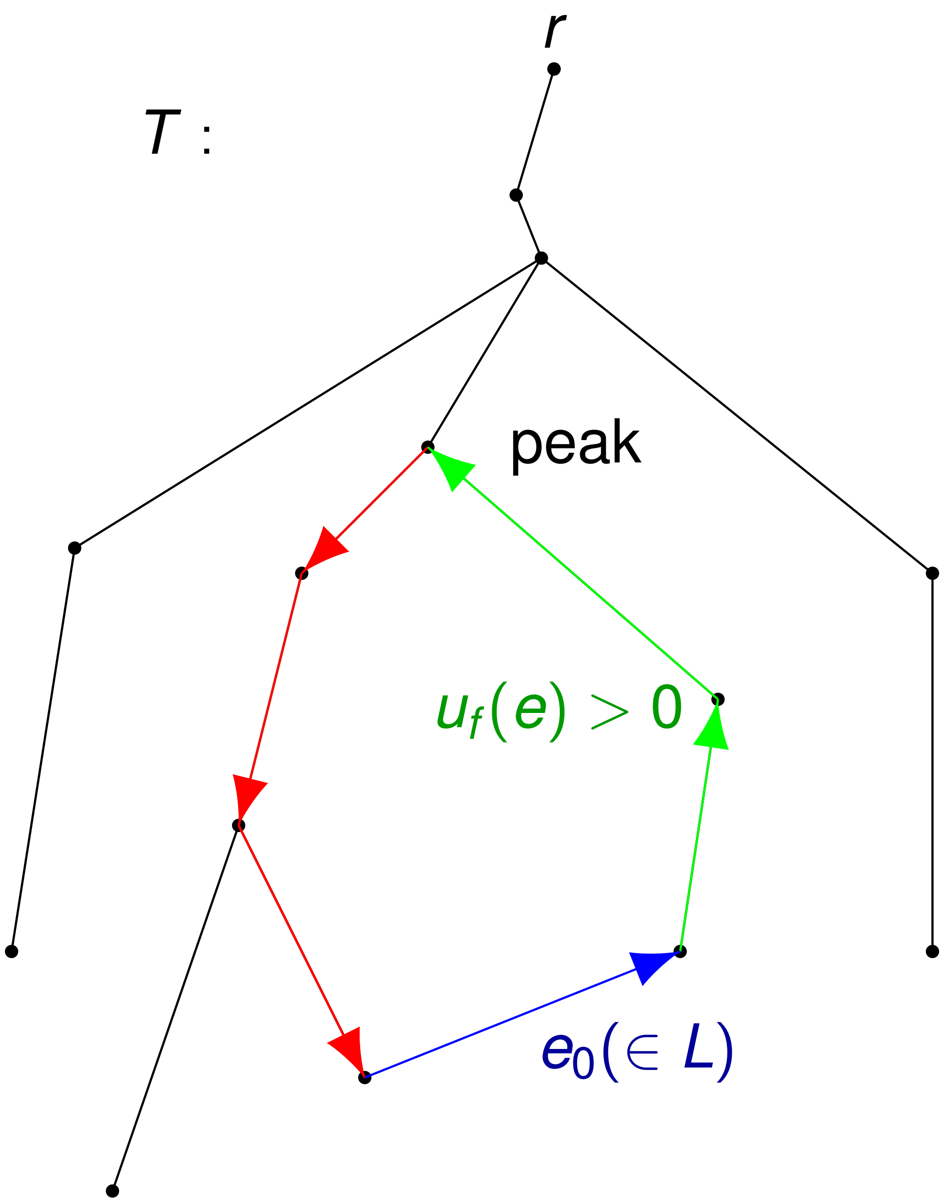
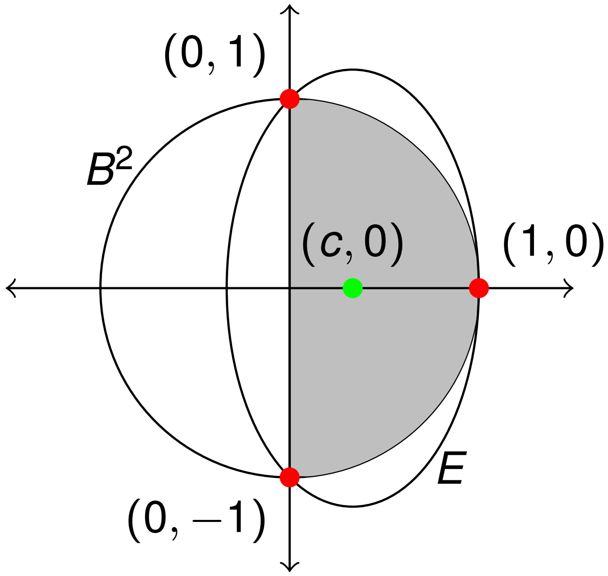


Illustration:



Cost of fundamental circuit $= c_\pi(e_0)$.



Half-Ball Lemma

$$B^n \cap \{x \in \mathbb{R}^n \mid x_1 \geq 0\} \quad \subseteq \quad E$$

with

$$E = \left\{ x \in \mathbb{R}^n \mid \frac{(n+1)^2}{n^2} \left(x_1 - \frac{1}{n+1} \right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1 \right\}.$$

Moreover, $\frac{\mathrm{vol}(E)}{\mathrm{vol}(B^n)} \leq e^{-\frac{1}{2(n+1)}}$.

Algorithm 5: Idealized Ellipsoid Algorithm

Input: A separation oracle for a closed convex set $K \subseteq \mathbb{R}^n$, a number $R > 0$ with $K \subseteq \{x \in \mathbb{R}^n \mid x^t x \leq R^2\}$, and a number $\epsilon > 0$.

Output: An $x \in K$ or the message "$\text{vol}(K) < \epsilon$".

$p_0 := 0$, $A_0 := R^2 I_n$;

for $k = 0, \ldots, N(R, \epsilon) := \lfloor 2(n+1)(n \ln(2R) + \ln(\frac{1}{\epsilon})) \rfloor$ **do**

  if $p_k \in K$ **then**

    return p_k;

  Let $\bar{a} \in \mathbb{R}^n$ be a vector with $\bar{a}^t y > \bar{a}^t p_k$ for all $y \in K$;

  $b_k := \frac{A_k \bar{a}}{\sqrt{\bar{a}^t A_k \bar{a}}}$;

  $p_{k+1} := p_k + \frac{1}{n+1} b_k$;

  $A_{k+1} := \frac{n^2}{n^2-1}(A_k - \frac{2}{n+1} b_k b_k^t)$;

return "$\text{vol}(K) < \epsilon$";

$\widetilde{p_k}$ and $\widetilde{A_k}$: exact values
p_k and A_k: rounded values

Adjust $\widetilde{A_k}$ by multiplying it by $\mu = 1 + \frac{1}{2n(n+1)}$.

$x \in K \Rightarrow$

- $(x - \widetilde{p}_k)^t \widetilde{A_k}^{-1} (x - \widetilde{p}_k) \quad \leq \quad 1 - \frac{1}{4n^2}$
- $(x - p_k)^t A_k^{-1} (x - p_k) \quad \leq \quad 1 - \frac{1}{4n^2} + 2\sqrt{n}\delta \ \|\widetilde{A_k}^{-1}\| \ (R + \|\widetilde{p_k}\|) + n\delta^2 \|\widetilde{A_k}^{-1}\| + (R + \|p_k\|)^2 \ \|A_k^{-1}\| \cdot \|\widetilde{A_k}^{-1}\| \cdot n\delta$

Goal is to choose δ such that

- $2\sqrt{n}\delta \ \|\widetilde{A_k}^{-1}\| \ (R + \|\widetilde{p_k}\|) + n\delta^2 \|\widetilde{A_k}^{-1}\| \ + \ (R + \|p_k\|)^2 \ \|A_k^{-1}\| \cdot \|\widetilde{A_k}^{-1}\| n\delta < \frac{1}{4n^2}$
- $\delta \|\widetilde{A_{k+1}}^{-1}\| < \frac{1}{4(n+1)^3}$

Proposition

Assume that $\delta \le \frac{1}{12n4^k}$ in iteration k of the ELLIPSOID METHOD. Then:

(a) A_k is positive definite.

(b) $\|p_k\| \le R2^k$, $\|\widetilde{p_k}\| \le R2^k$.

(c) $\|A_k\| \le R^2 2^k$, $\|\widetilde{A_k}\| \le R^2 2^k$.

(d) $\|A_k^{-1}\| \le R^{-2} 4^k$, $\|\widetilde{A_k}^{-1}\| \le R^{-2} 4^k$.

Algorithm 6: Ellipsoid Algorithm

Input: A separation oracle for a closed convex set $K \subseteq \mathbb{R}^n$, a number $R > 0$ with $K \subseteq \{x \in \mathbb{R}^n \mid x^t x \leq R^2\}$, and a number $\epsilon > 0$

Output: An $x \in K$ or the message "$\text{vol}(K) < \epsilon$".

$p_0 := 0$, $A_0 := R^2 I_n$;

for $k = 0, \ldots, N(R,\epsilon) := \lceil 8(n+1)(n \ln(2R) + \ln(\frac{1}{\epsilon})) \rceil$ **do**

  if $p_k \in K$ **then**

    return p_k;

  Let $\bar{a} \in \mathbb{R}^n$ be a vector with $\bar{a}^t y > \bar{a}^t p_k$ for all $y \in K$;

  $b_k := \frac{A_k \bar{a}}{\sqrt{\bar{a}^t A_k \bar{a}}}$;

  p_{k+1} an approximation of $\widetilde{p_{k+1}} := p_k + \frac{1}{n+1} b_k$ with maximum error $\delta < \left(2^{6(N(R,\epsilon)+1)} 16 n^3\right)^{-1}$;

  A_{k+1} a symmetric approximation of $\widetilde{A_{k+1}} := \left(1 + \frac{1}{2n(n+1)}\right) \frac{n^2}{n^2-1} (A_k - \frac{2}{n+1} b_k b_k^t)$ with maximum error δ;

return "$\text{vol}(K) < \epsilon$";

Theorem

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$. Then, the following statements are equivalent:

(a) P is integral

(b) Each face of P contains at least one integral vector.

(c) Each minimal face of P contains at least one integral vector.

(d) Each supporting hyperplane of P contains at least one integral vector.

(e) Each rational supporting hyperplane of P contains at least one integral vector.

(f) $\max\{c^t x \mid x \in P\}$ is attained by an integral vector for each c for which the maximum is finite.

(g) $\max\{c^t x \mid x \in P\}$ is an integer for each integral vector c for which the maximum is finite.

Theorem

A matrix $A = (a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \in \mathbb{Z}^{m\times n}$ is totally unimodular if and only if for each set $R \subseteq \{1,\ldots,n\}$ there is a partition $R = R_1 \dot\cup R_2$ such that for each $i \in \{1,\ldots,m\}$: $\sum_{j\in R_1} a_{ij} - \sum_{j\in R_2} a_{ij} \in \{-1,0,1\}$.

The **incidence matrix** of an undirected graph G is the matrix $A_G = (a_{v,e})_{\substack{v\in V(G) \\ e\in E(G)}}$ which is defined by:

$$a_{v,e} = \begin{cases} 1, & \text{if } v \in e \\ 0, & \text{if } v \notin e \end{cases}$$

The **incidence matrix** of a directed graph G is the matrix $A_G = (a_{v,e})_{\substack{v\in V(G) \\ e\in E(G)}}$ which is defined by:

$$a_{v,(x,y)} = \begin{cases} -1, & \text{if } v = x \\ 1, & \text{if } v = y \\ 0, & \text{if } v \notin \{x,y\} \end{cases}$$

Definition

Let $P \subseteq \mathbb{R}^n$ be a convex set. Let M be the set of all rational half-spaces $H = \{x \in \mathbb{R}^n \mid c^t x \leq \delta\}$ with $P \subseteq H$. Then, we define

$$P' := \bigcap_{H \in M} H_I.$$

We set $P^{(0)} := P$ and $P^{(i+1)} := \left(P^{(i)}\right)'$ for $i \in \mathbb{N} \setminus \{0\}$. $P^{(i)}$ is the i-th **Gomory-Chvátal-truncation** of P.

Lemma

Let $H = \{x \in \mathbb{R}^n \mid c^t x \leq \delta\}$ be a rational half-space such that the components of c are relatively prime integers. Then $H_I = H' = \{x \in \mathbb{R}^n \mid c^t x \leq \lfloor \delta \rfloor\}$.