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Exercise Set 4

Exercise 4.1. For a finite set $\emptyset \neq T \subsetneq \mathbb{R}^2$ we define

$$\mathrm{BB}(T) := \max_{(x,y)\in T} x - \min_{(x,y)\in T} x + \max_{(x,y)\in T} y - \min_{(x,y)\in T} y.$$

A *Steiner tree* for T is a tree Y with $T \subseteq V(Y) \subsetneq \mathbb{R}^2$. We denote by $\mathrm{Steiner}(T)$ the length of a shortest rectilinear (i.e. edge lengths acc. to ℓ_1) Steiner tree for T. Moreover let $\mathrm{MST}(T)$ be the length of a minimum spanning tree in the complete graph on T with edge costs ℓ_1.

Prove that:

(a) $\mathrm{BB}(T) \leq \mathrm{Steiner}(T) \leq \mathrm{MST}(T)$;

(b) $\mathrm{Steiner}(T) \leq \frac{3}{2}\,\mathrm{BB}(T)$ for $|T| \leq 5$;

(c) There is no $\alpha \in \mathbb{R}$ s.t. $\mathrm{Steiner}(T) \leq \alpha\,\mathrm{BB}(T)$ for all finite $\emptyset \neq T \subset \mathbb{R}^2$.

(2 + 3 + 2 points)

Exercise 4.2. Let T be an instance of the Rectilinear Steiner Tree Problem and $r \in T$. For a rectilinear Steiner tree Y we denote by $f(Y)$ the maximum length of a path from r to any element of $T \setminus \{r\}$ in Y.

(a) Find an instance where no Steiner tree minimizes both length and f.

(b) Consider the problem of finding a shortest Steiner tree Y minimizing $f(Y)$ among all shortest Steiner trees. Is there always a tree with these properties which is a subgraph of the Hanan grid?

(1 + 4 points)

Exercise 4.3. Consider the following algorithm to compute a rectilinear Steiner tree T for a set P of points in the plane $\mathbb{R}^2$.

In this notation $SP(u,w) \subset \mathbb{R}^2$ is the area covered by shortest paths between u and w, and $dist(s,T)$ is the minimum distance between s and the shortest path area $SP(u,w)$ of an edge $\{u,w\} \in E(T)$.

Choose $p \in P$ arbitrarily;
$T := (\{p\}, \emptyset), S := P \setminus \{p\}$
while $S \neq \emptyset$ **do**
    Choose $s \in S$ with minimum $dist(s, T)$
    Let $\{u, w\} \in E(T)$ be an edge which minimizes $dist(s, SP(u, w))$
    $v := \arg\min\{dist(s, v) \mid v \in SP(u, w)\}$
    $T := (V(T) \cup \{v\} \cup \{s\}, E(T) \setminus (u, w) \cup \{u, v\} \cup \{v, w\} \cup \{v, s\}\}$
    $S := S \setminus \{s\}$
end while

Show that the algorithm is a $\frac{3}{2}$-approximation algorithm for the Minimum Steiner Tree Problem.

Hint: First show that the length of T is at most the length of a minimum spanning tree on P.

(8 points)

Deadline: **Tuesday**, May 15th, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss18/chipss18.html`

In case of any questions feel free to contact me at bihler@or.uni-bonn.de.