Bonn Problem-Solving Seminar, BPS 2. July 26, 2013

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(In the solutions, it is feasible to rely on such basic results as Kőnig theorem, Menger theorem, MFMC theorem and algorithm, Dijkstra algorithm, etc.)

1. It is known that a cheapest st-path in a digraph $D = (V, A)$ with a non-negative cost function on A can be efficiently computed with the help of the Dijkstra algorithm. Develop a polynomial time algorithm to decide if D includes k edge-disjoint cheapest st-paths.

2. A hypergraph $H = (V, \mathcal{E})$ is called $(1, 1)$-partition-connected if, for each partition $\mathcal{P}$ of V with $|\mathcal{P}| \geq 2$ there are at least $|\mathcal{P}|$ hyperedges intersecting at least two members of $\mathcal{P}$. **(A)** Develop a polynomial algorithm to decide if a hypergraph is $(1, 1)$-partition-connected. **(B)** Decide whether it is true or not that a graph G is $(1, 1)$-partition-connected if and only if G is 2-edge-connected.

3. Let ab and cd be edges of a simple undirected graph $G = (V, E)$ so that ac and bd are not edges of G. By an elementary change (with respect to G) we mean the operation of replacing the existing edges ab and cd by the new edges ac and bd. Clearly, the resulting graph G' is also simple and admits the same degree sequence as G does. **(A)** Prove that it is possible to get any graph $G' = (V, E')$ with the same degree sequence from G by a series of elementary changes (where an elementary change always concerns the current graph). **(B)** Find an upper bound for the number of necessary elementary changes and develop a polynomial time algorithm for constructing the transition from G to G' by elementary changes.

4. Let $G = (S, T; E)$ be a bipartite graph. **(A)** Prove that there are two disjoint subsets K and N of edges such that $d_N(v) = d_K(v)+1$ for every node v of G if and only if $|S| = |T|$ and $d_G(X) \geq ||X \cap S|-|X \cap T||$ holds for every subset $X \subseteq S \cup T$. **(B)** Construct an example to demonstrate that the following necessary condition is not sufficient in general: $|\Gamma(X)| + d(\Gamma(X), S - X)) \geq |X|$ holds for every $X \subseteq S$ where $d(A, B)$ denotes the number of edges connecting $A - B$ and $B - A$. **(C)** Develop an algorithm to find K and N.

5. Let $D = (V, A)$ be a digraph with a root-node r_0 and assume that the underlying undirected graph is connected. As long as possible select an arbitrary dicut B (in the current digraph) that is oriented toward r_0 and reorient B (that is, reverse the orientation of each edge in B). **(A)** Prove that after a finite number of dicut reorientations the resulting digraph is root-connected. **(B)** Prove that after a polynomial number of dicut reorientations the resulting digraph is root-connected. **(C)** Prove that the final digraph is independent of the intermediate choices of dicuts.

6. Let G be an undirected graph. **(A)** Prove that if G is not bipartite, then every strongly connected orientation of G includes a di-circuit of odd length. **(B)** Prove that if G inculdes an odd cut, then every acyclic orientation of G includes a dicut of odd cardinality.

7. Every edge of a digraph $D = (V, A)$ is coloured by red and/or blue in such a way that, for every pair $\{u, v\}$ of nodes, there is a red or a blue uv-dipath or vu-dipath. Prove that there is a node r of D so that there is a red or blue ru-dipath for every node u.

8. Let $D = (V, A)$ be a digraph. A subset $B \subseteq A$ of edges is **circuit-equitable** if for every circuit C of D (in the undirected sense) the number of B-edges in one direction along C is the same as the number of B-edges in the other direction. Design an efficient algorithm to decide if a given B is circuit-equitable.

9. Let $D = (V, A)$ be a strongly connected digraph with $|V| \geq 3$ and let $Z \subseteq V$ be a subset of nodes inducing a tournament. Prove that there is a di-circuit of D covering every element of Z.

10. Let D denote the digraph arising from a bipartite graph $G = (S, T; E)$ by orienting each edge of G toward T. Prove that the maximum number of disjoint cuts of G is the same as the maximum number of disjoint dicuts of D.